MTH 605: Topology I

Practice Assignment V

1. A space X is *contractible* if the identity map on X is nullhomotopic.
   (a) Show that $\mathbb{R}^n$ is contractible.
   (b) Show that any contractible space is path connected.
2. Let x_0 and x_1 be points in a path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair f and g of paths from x_0 to x_1, we have $\hat{f} = \hat{g}$.
3. For $A \subset X$, a continuous map $r : X \to A$ such that $r|_A = i_A$ is called a *retraction* of X into A. If $a \in A$, show that $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.
4. Let A be a subspace of $\mathbb{R}^n$, and let $h : (A, a) \to (Y, y)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is trivial.
5. Let $p : \widetilde{X} \to X$ be continuous and surjective. Suppose that U is open set in X that is evenly covered by p. Then show that if U is connected, then the partition of $p^{-1}(U)$ to slices is unique.
6. Let $p : \widetilde{X} \to X$ be a covering map, and let X be connected. Show that if $p^{-1}(x_0)$ has k elements for some $x_0 \in X$, then $p^{-1}(x)$ has k elements for every $x \in X$. In such a case, $\widetilde{X}$ is an *k-fold covering space* of X.
7. Show that $p_n : S^1(\subset \mathbb{C}) \to S^1(\subset \mathbb{C})$ given by $p_n(z) = z^n$ is an n-fold covering space for every positive integer n.
8. Describe all k-fold covering spaces of the torus $S^1 \times S^1$.